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Production economics and process quality: A Taguchi perspective

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Abstract

Although several studies have analyzed the interaction between the economics of production and process quality, most of them view quality from a very traditional perspective – reject when outside specified limits, or else accept. Recent views on quality have shown that such a definition greatly underestimates the costs of poor quality and leads to sub-optimal decisions. The primary intent in this paper is to revisit this interaction of the economics of production with process quality from a non-traditional yet more realistic "Taguchi" quality cost perspective. Specifically, we investigate the possibility of investing in a process to decrease its variance. Although such an investment reduces the proportion of defects, and when large enough, the Taguchi's loss, it also increases the cost of holding inventory. Our model determines the optimal levels of inventory, and the production lot-size that minimizes the sum of inventory and quality-related costs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Over the last decade, a stream of research has examined the relationship between the economics of production and quality. One insight from this research is that investing in the production process to either reduce its setup costs (and time), or improve its reliability results in higher process quality levels. As a consequence, practicing operations managers can improve the quality levels in their process by producing smaller batches while maintaining the same level of inventory-quality system costs. Such results seem consistent with the

Japanese production systems that have over the last two decades excelled with the use of smaller lot-sizes and continual process improvement. Our primary intent here is to revisit this interaction of the economics of production with process quality, however, from a non-traditional yet more realistic "Taguchi" quality cost perspective.

Porteus [1] was perhaps the first to explicitly model the relationship between quality and the economics of production. He assumed that a process can go "out of control" with a given probability with each unit produced. This assumption would justify smaller lot-sizes, since it leads to fewer defective items. In this seminal paper, he also introduced the option of investing in the process to reduce the probability that the process goes out of control. This leads to a bi-variate decision model,

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where the lot-size and investment are jointly determined. Instead of a one-time investment, Fine and Porteus [2] later refine the above model to allow smaller investments over time with potential process improvements of random magnitude.

Several researchers have since extended these two initial studies. Notable among them are Chand [3], who validates Porteus' [1] model when learning effects are present. Larson [4] shows that the lot-sizes and the total inventory-quality costs go down as a firm invests in vendor relations and quality improvement programs. In a series of papers, Cheng [5,6] extends the classic economic production quantity (EPQ) (see [7]) problem by assuming, first, that the production costs are a function of the reliability (proportion of defectives) and demand, and second, that they are a function of reliability and setup costs, respectively.

Although all the above studies describing the interaction between the economics of production and investment in process quality show considerable promise for increased quality with reduced overall system costs, they have one major shortcoming. From a conceptual point of view, all published research in this area view quality from a traditional perspective – reject if the quality characteristic in question is outside specification limits, or else, accept.

However, recent studies in quality (see for example [8]) have shown that the traditional definition of failures greatly underestimate the quality costs. A classic example (adapted from Taguchi and Clausing [8, p. 67]) that illustrates this point is the Ford versus Mazda case. Ford, in addition to building transmissions itself, had also asked Mazda to build transmissions to the same specifications for a car they planned to sell in the US. It was noticed after some time that the cars with Ford's transmissions were generating far more complaints and resulting in higher warranty costs. On closer inspection, it was observed that Ford had high variability in their gearboxes (several units close to the specification limits) while Mazda's gearboxes were close to target. Although Ford was working within the definition of zero defects, they incurred a higher cost.

From his experience, Taguchi characterizes this cost or loss as a quadratic function. This loss reduc-

es to zero, when the production process manufactures at exactly the *target value* (determined independently), and it increases quadratically as the process moves away from the mean. If x is the actual value of the quality characteristic, Taguchi defines the loss per unit, $L(x)$, for the products that have been shipped as

$$L(x) = K(x - \mu)^2, \qquad (1)$$

where μ is the target value of the production process, and K can be defined as the loss per unit for a unit deviation from the mean. The loss $L(x)$ estimates the cost to 'society' from the failure of the product to meet its target value for a given quality characteristic. The loss can be incurred by the customer as maintenance or repair costs; by the manufacturer as warranty or scrap costs; or by the society in general, as pollution or environmental costs.

Since all the studies reviewed above view quality from a very traditional perspective, they consistently underestimate the cost of poor quality leading to *suboptimal decisions* [9]. Although we address issues that are similar to above studies, our primary contribution is that our model explicitly accounts for the loss that is incurred even when the products are made to specifications. To the best of our knowledge, our paper is the first to link lot-size determination to the loss-based quality accounting systems.

2. Environment and notation

In this paper we consider the classic EPQ model (see [7]) without shortages or backorders, and assume that a process produces a single product in batch sizes of Q. The demand rate (D) for the product is deterministic and constant over a planning horizon of one year. The production process produces this product with a finite production rate (P). The quality characteristic of interest, X, is assumed to be a random variable that is symmetric around the mean that is set at the 'target value' (μ) i.e., we assume that the process on average produces the right mean, and a variance ($\sigma^2(I)$) that is a function of the investment (I) in the process.
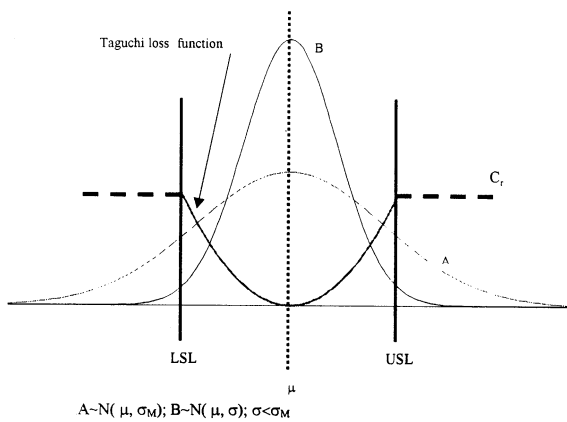
Fig. 1. The model environment.

A product is considered defective if the quality characteristic, X, lies outside the specification limits. These defective items are rejected with a cost. It is assumed that the detection of defectives is achieved by non-destructive and error-free testing. Fig. 1 illustrates this environment, where X follows a normal distribution. Curve A is the production process in its current state (with a mean μ and a variance, say σ_M^2). The items that fall within the specification limits are held in inventory (with a cost) for future shipment to the customer. As the ensuing discussion will show, every item that is shipped to the customer is also accessed a "Taguchi loss cost" that is dependent on the parameters of the loss function and the production process-distribution. All items that fall outside the specification limits are rejected with a cost, C_r.

When an investment is made in the process, the variance of the production process decreases to a value $\sigma^2(I) \leqslant \sigma_M^2$ (curve B) which will decrease the number of defective items, and hence the reject costs. The Taguchi losses will either increase or decrease, depending on the size of the investment. For a given lot size, investing in the process will also increase the cost of holding inventory since on an average more items are held in a production cycle.

On the other hand, for a given level of investment, an increase in the production lot-size increases inventory but decreases set-up costs. Our primary objective, therefore, is to find the right level of investment and production lot-size that optimizes the expected inventory-quality system costs. The following common notation is used:

D	demand per day
R	annual demand = $250D$, assuming 250 working days a year
P	production rate per day
Q	lot size
h	holding cost/unit/yr
A	setup cost
USL, LSL	upper, lower specification limits
I	process investment
$p(I)$	proportion of defectives
C_r	reject cost/unit
K	'Taguchi's' loss parameter, a constant that is defined as the loss incurred per unit for a unit deviation from the mean
μ	process mean, assumed to be the same as the target value
$\sigma^2(I)$	process variance as a function of investment. When $I = 0$, $\sigma^2(I) = \sigma_M^2$ and when $I = \infty$, $\sigma^2(I) = \sigma_L^2$.
C_{pk}	process capability index defined as $\Delta/3\sigma(I)$, where $\Delta = (\text{USL} - \mu) = (\mu - \text{LSL})$
$f_I(\cdot)$	probability distribution function (pdf) of the manufacturing process distribution $\sim \text{N}(\mu, \sigma^2(I))$
$\psi_I(\cdot)$	cumulative distribution function (cdf) of the above manufacturing process distribution

3. The basic model and observations

3.1. The model

Consider a firm having an expected total annual cost (ETAC) as follows:

$$\text{ETAC}(Q, I) = [[R/(1 - p(I))]/Q]A$$
$$+ Q(1 - p(I))h\theta/2$$
$$+ [R/(1 - p(I))]p(I)C_r$$
$$+ E[L(X)]R + I. \qquad (2)$$

The first term of the cost equation is the setup costs incurred by the firm in a year. The total number of units produced in a year is $R/(1 - p(I))$, therefore the number of setups in a year is $[R/(1 - p(I))]/Q$. The second term is the holding costs. Since $Qp(I)$ units are rejected, each cycle produces only $Q(1 - p(I))$ items that can be held in inventory (the rest are rejected at a cost). Since inventory is gradually replenished during the production cycle, the maximum inventory in any given production cycle can be calculated as $(1 - p(I))Q\theta$ where θ is defined as $(P - D)/P$. The average rate of inventory build up therefore is $(1 - p(I))Q\theta/2$. To make the model non-trivial, it is assumed that $0 \leqslant \theta \leqslant 1$, i.e., the production rate is at least the demand rate for our EPQ model. The third term in the equation is the reject cost. An average of $[R/(1 - p(I))]p(I)$ units are rejected every year, with a cost C_r. The fourth term is the total 'Taguchi loss' costs. The expected Taguchi's loss, $E[L(X)]$, can be written as

$$K \int_{LSL}^{USL} (x - \mu)^2 f_I(x)\, dx. \tag{3}$$

We define (see also [10]) K as C_r/Δ^2, where Δ is the distance of the mean from the specification limits. When $x = \mu$, the loss is zero, and when $x = LSL$ or USL, the loss is equal to the reject loss C_r. Therefore, the loss increases quadratically from 0 to C_r, as the quality characteristic moves away from the intended mean.

To facilitate the analysis, it is assumed here that the upper and lower specification limits are equidistant from the mean i.e., $\Delta = (USL - \mu) = (\mu - LSL)$. Since a total of R units are shipped every year, the total Taguchi's loss for the year is $E[L(X)]R$. The overall objective is, of course, to find the values of Q and I that minimize the total inventory-quality system cost represented in Eq. (2).

Additionally, we represent the variance of the process, $\sigma^2(I)$, as a function of investment (see also [11]):

$$\sigma^2(I) = \sigma_L^2 + (\sigma_M^2 - \sigma_L^2)\exp(-bI), \quad b > 0. \tag{4}$$

σ_M^2 is the maximum (or current) level of the variance of the system, and σ_L^2 is the minimum level to which the process variance can be decreased. We note here that the above function is only one representation of a variance reducing function. It was the most logical choice for us since (i) the variance has an upper and a positive lower bound, and (ii) the marginal value of investment (w.r.t. variance) decreases as one invests more into the process. One could, however, envisage many other polynomial forms for the variance reduction equation. In the following sections, however, our analysis *will assume* that $\sigma^2(I)$ follows the form in Eq. (4).

3.2. Observations and analysis

We illustrate our analysis assuming that the production process follows a Normal distribution. Using this representative distribution, our objectives in this section are to carefully study the structure of the cost function in order to gain insights into its behavior with respect to both investment and lot-size. For the purposes of brevity, we omit the proofs of the following propositions. The proofs are, however, available from the authors.

The pdf of the normal distribution with mean μ and variance $\sigma^2(I)$ is

$$\frac{1}{\sqrt{2\pi}\sigma(I)} e^{-(x-\mu)^2/2\sigma^2(I)}. \tag{5}$$

Proposition 1. (a) *The proportion of defectives, $p(I)$, in the production process is bounded (with bounds that are different from 0 or 1) and are given by*

$$2\Phi(-\Delta/\sigma_L) \leqslant p(I) \leqslant 2\Phi(-\Delta/\sigma_M),$$

where $\Phi(\cdot)$ is the cdf of the standard normal.

(b) *The Process Capability Index or $C_{pk} > 0.471$ is a sufficient condition to show that $p(I)$ is strictly convex w.r.t. I.*

Proposition 1(a) shows that the fraction of defectives are bounded (with bounds that are different from 0 or 1). A bound on $p(I)$ establishes a bound

on the reject costs too. We later show that the Taguchi's loss is also bounded as I increases The key insight here is that investment in the process would probably not be justified once the losses are close to their lower bounds.

Proposition 1(b) describes a *sufficient* condition under which $p(I)$ is *strictly* convex. $C_{pk} = \Delta/3\sigma(I)$ is a popular index that managers use to judge how reliable a process is. A C_{pk} of 1.5 translates to a defect rate of 3.4 parts per million (ppm) or Motorola's famous 'Six-Sigma Quality'. The condition $C_{pk} > 0.471$ translates to a production process producing a defect rate of 160,000 ppm or less. We later use this result to prove the convexity of the reject costs, and ultimately the convexity of *ETAC* w.r.t. I.

Proposition 2. (a) *The expected Taguchi's loss is given by* $2C_rR/9C_{pk}^2(1/2 + s\phi(s) - \Phi(s))$, *where* $s = -3C_{pk}$. *This loss increases as long as* $I \leqslant -(1/b)\ln/(\Delta^2/1.88 - \sigma_L^2)/(\sigma_M^2 - \sigma_L^2)$ *and then decreases monotonically.*

(b) *As I increases, the expected Taguchi's loss approaches a (lower) bound given by* $2C_rR\sigma_L^2/\Delta^2(1/2 + \Delta/\sigma_L\phi(\Delta/\sigma_L) - \Phi(\Delta/\sigma_L))$.

Proposition 2(a) describes the behavior of the yearly expected Taguchi loss with increasing investment in the process. We find that it initially increases, but after a certain point (recall that this point is specific to the form of $\sigma(I)$ in (4)) it starts to decrease (see also [12]). The direct implication of the proposition is that the maximum benefit from investment in the process is derived beyond this turning point.

Proposition 2(b) meanwhile establishes a lower bound on the yearly Taguchi's loss implying that the marginal benefit (in reduced costs) of investment decreases as the loss approaches its lower bound.

Proposition 3. (a) *For a given lot-size, Q_0 the step-up cost is a non-increasing function of I.*

(b) *For a given lot-size, Q_0, the holding cost is a non-decreasing function of I.*

(c) *ETAC is convex with respect to Q, and for a given level of investment, I, the optimal lot-size takes*

the form

$$Q(I) = \frac{1}{(1 - p(I))}\sqrt{\frac{2AR}{h\theta}}.$$

(d) *When the lot-size takes the form in proposition 3(c), the ordering and the holding costs are equal and constant given by* $\sqrt{RAh\theta/2}$.

The above set of propositions show the behavior of the inventory elements of the cost function w.r.t. to I and Q. The form of the optimal lot-size from (c) is similar to the classic EPQ model – only it is inflated by a fraction $1/(1 - p(I))$ to account for the defective items produced every cycle. Additionally, when we substitute the form of the optimal lot-size, both the set-up and the holding costs, like the classic EPQ model, are equal and are reduced to a constant. Furthermore, from Proposition 3(c), we also know that the ETAC is convex in Q. Consequently, we can easily calculate the bounds for the optimal lot-size:

$$Q_{max} = \frac{1}{(1 - p_{max})}\sqrt{\frac{2AR}{h\theta}} \, Q_{opt} \geqslant Q_{min}$$

$$= \frac{1}{(1 - p_{min})}\sqrt{\frac{2AR}{h\theta}}.$$

Out task therefore is to find the level of investments, I, in the above range that minimizes the ETAC. The next set of proposition determine the shape of the ETAC w.r.t. I, when $Q_{max} \geqslant Q \geqslant Q_{min}$.

Proposition 4. (a) *When $P(I)$ is convex,* i.e., *when $C_{pk} > 0.471$, the reject cost is convex function of I*

(b) $C_{pk} > 0.251$ *is a sufficient condition for the sum total of the Taguchi's loss and the reject cost to be a strictly convex function of I.*

(c) *When the lot-size assumes the form in proposition 3(c), and when $C_{pk} > 0.471$, there is a unique I that minimizes ETAC.*

We had earlier established that ETAC was convex in Q. Proposition 3(c) which results from Proposition 3(a) and 3(b), proves that when $C_{pk} > 0.471$, ETAC is strictly convex in I. Therefore, there exists an unique Q_{opt} and I_{opt} that minimizes ETAC. Simple numerical techniques such as the Newton's

Table 1
Input parameters and results

Inputs		
Demand rate	D	100
Production rate	P	105
Ordering cost	A	$100
Holding cost/unit/yr	h	$20
Lower specification limit	LSL	4.8
Upper specification limit	USL	5.2
Reject cost	C_r	$1.00
Process mean	μ	5
Initial maximum) Process variance	σ_M^2	0.1
Minimum process variance	σ_M^2	0.005
Variance curve constant	b	0.0035
Derived values		
Inventory buildup rate	θ	0.0476
Taguchi constant	K	$25.00
Proportion defectives	p	0.00693
Variance with investment	$\sigma^2(I)$	0.00549
Annual demand	R	25 000
Decision variables		
Lot-size	Q	2307.28
Investment	I	$1,506.98
Costs		
Investment	I	$1,506.98
Ordering		$1.091.09
Holding		$1,901.09
Reject		$174.50
Taguchi's loss		$3,212.38
Total		$7,076.03

or the Conjugate-Gradient method can be used to find Q_{opt} and I_{opt}.

4. Numerical illustration

In this section, we illustrate the model through a numerical illustration. Assume that a firm needs to manufacture widgets of length 5 in (μ). The production process is a normal distribution, with mean (μ) of 5 in, and a variance (i.e., the current level, σ_M^2), 0.1 in. As investment, I, can decrease this variance to a minimum of 0.005 in (σ_L^2). The USL and the LSL are 5.2 and 4.8 in respectively. Any widget that falls outside this specification limit is rejected with a cost ($C_r = \$1$). The production rate ($P$) is maintained at 105 units/day to satisfy the demand rate

(D) of 100 units/day. The investment-variance relation is assumed to be the following:

$$\sigma^2(I) = 0.005 + 0.095e^{-0.0035I}.$$

The firm, in this case, has two decisions to make — what lot-size to produce (Q), and how much to invest (I) in the process. Table 1 outlines all the other input parameter levels. Fig. 2 illustrates how ETAC varies with both Q and I. The convexity is obvious from the figure, and the cost-minimizing values of the lot-size and investment (obtained through Newton's method) are $Q_{opt} = 2307.28$ and $I_{opt} = \$1506.98$. The breakdown of the various costs involved are also shown in Table 1.

Table 1 also indicates that the optimal proportion of defectives is 0.0069, just above its lower bound (From Proposition 1) of 0.0046. In this example, $\sigma^2(I)$ is decreased from 0.1 ($C_{pk} = 0.2108$) to 0.0055 ($C_{pk} = 0.9$) just above its lower bound of 0.005.

The optimal levels of investment and the lot-size are dependent on the parameter b, that determines how quickly $\sigma^2(I)$ reaches its lower bound. As Fig. 3 illustrates, the optimal lot-size and the optimal investment decrease with increasing b. It is easy to see that with a higher value of b, a lesser amount of investment is needed to reduce levels of $\sigma^2(I), p(I)$, reject costs, and Taguchi's looses to the desired levels. Additionally, from the form of the optimal lot-size in Proposition 3(c), increasing b for a fixed amount of investment, I, decreases the lot-size. From a managerial perspective, the parameter, b might represent the methods used to improve the process. A lower b can correspond to, say, a traditional tune-up of an old machine, while a higher b to a more innovative way that produces a higher level of improvement while costing the same as a tune-up.

Finally, if we solve this firm's problem *without* the Taguchi's losses, the optimal values of lot-size and investment are 2322.75 and $1173.81, respectively. $\sigma^2(I)$ is decreased from 0.1 ($C_{pk} = 0.2108$) to 0.0066 ($C_{pk} = 0.823$). We note that the firm, in this case, has underestimated the cost of quality, and consequently invests *less* to improve the process. An investment of $1173.81 and a lot-size of 2322.75 translates to ordering and holding costs of $1091.09, a reject cost of $343.31, and a Taguchi

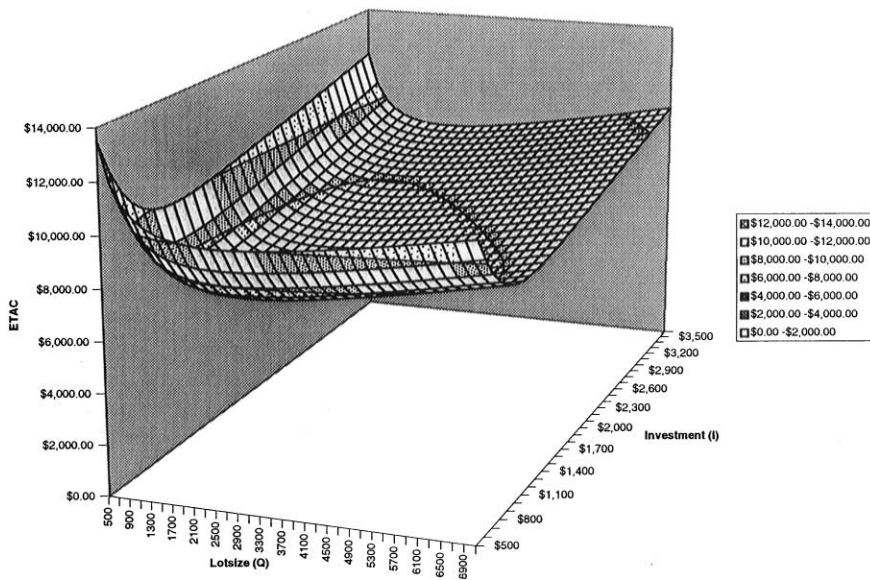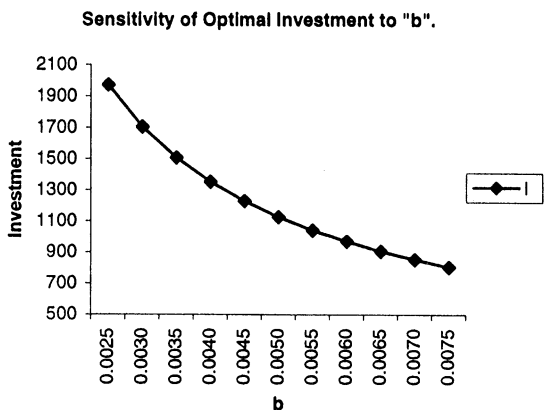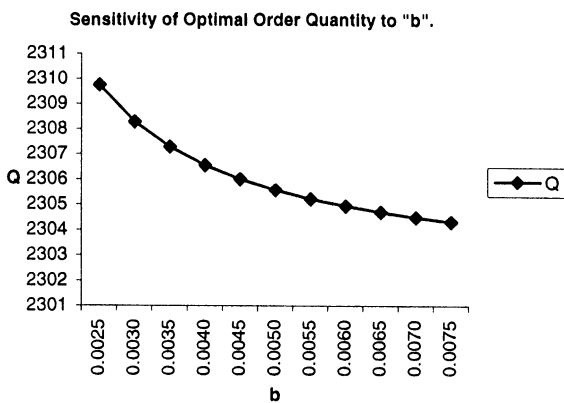Fig. 2. ETAC vs. Q and I.

Fig. 3. Sensitivity of optimal results to the parameter b.

loss of $3661.98, yielding a total cost of $7,361.28, over 4% higher than the total cost level in Table 1. This example is an indication that a solution that underestimates the costs of quality (but ignoring the Taguchi's losses) ultimately results in higher total costs.

5. Conclusions

Our primary objective in this paper was to study the interaction between the economics of production, and process quality. Our primary contribution comes from the fact our optimal solution includes the effect of the estimated Taguchi's losses. Such an explicit consideration of the inventory and quality costs has several managerial implications. First, it can be useful in benchmarking the inventory-quality cost. Managers can calculate the inventory-quality associated costs, and compare it to the optimal values given by the model. Steps can then be taken to improve the process to match these optimal levels. These steps are of course, investing in the process to decrease the variance, and producing the batch size recommended by the model. Second, the model gives a tool for managers to compare different production process.

Furthermore, the above analysis can also guide managers toward better choices for process improvement. For example, improving a certain process could involve either overhauling a machine, retooling it, or may be even buying a new one. Each one of these options will have a production distribution associated with it – and consequently a model such as the one presented in this paper can be used to choose the option that has the least inventory-quality costs.

We see future research primarily in two areas. First, how the model we have presented changes with different variance-investment functions, and second, the impact of using several quality characteristics, as opposed to just one, on our model.

Acknowledgements

References

[1] E.L. Porteus, Optimal lot-sizing process quality improvement and setup cost reduction, Operations Research 34 (1986) 137–144.

[2] C.H. Fine, E.L. Porteus, Dynamic process improvement, Operations Research 37 (1989) 580–591.

[3] S. Chand, Lot sizes and setup frequency with learning in setups and process quality, European Journal of Operations Research 42 (1989) 190–202.

[4] P.D. Larson, An integration of inventory and quality decisions in logistics: An analytical approach, Journal of Business Logisticsm 10 (1989) 106–122.

[5] T.C.E. Cheng, An economic production quantity model with flexibility and reliability considerations, European Journal of Operations Research 39 (1989) 174–179.

[6] T.C.E. Cheng, Economic order quantity model with demand-dependent unit production cost and imperfect production processes, IIE Transactions 23 (1991) 23–28.

[7] J.O. McClain, L.J. Thomas, Operations Management, Prentice-Hall, Englewood Cliffs, NJ, 1985.

[8] G. Taguchi, D. Clausing, Robust quality, Harvard Business Review 68 (1990) 65–75.

[9] C.D. Ittner, Exploratory evidence on the behavior of quality costs, Operations Research 44 (1996) 114–130.

[10] G. Taguchi, Y. Wu, Introduction to off-line quality control, Central Japan Quality Control Association, 1985, pp. 1–25.

[11] J. Hong, S.H. Xu, J.C. Hayya, Process quality improvement and setup reduction in dynamic lot-sizing, International Journal of Production Research 31 (1993) 2693–2708.

[12] S.K. Mukhopadhyay, D. Chakraborty, Optimal process variance under Taguchi loss, International Journal of Quality and Reliability Management 12 (1995) 14–29.